Entropy Numbers from Time-Frequency Representations

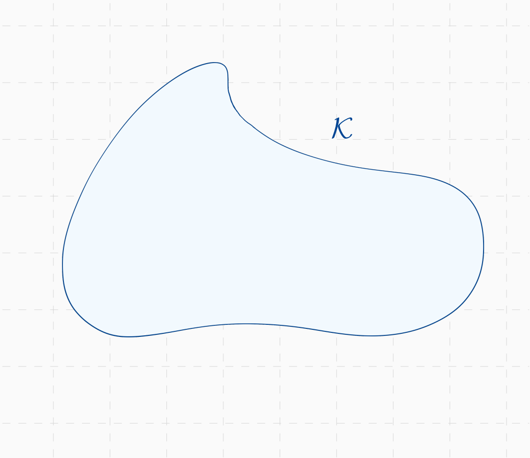
International Conference on Computational Harmonic Analysis 2026

Thomas Allard

Stanford University

May 22, 2026

Metric Entropy and Entropy Numbers



Covering Number.

$N(\varepsilon; \mathcal{K})$ = number of
to cover $\mathcal{K}$

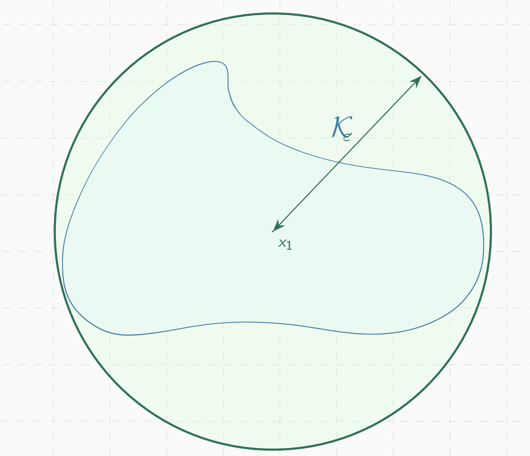
Metric Entropy.

$H(\varepsilon; \mathcal{K}) = \log(N(\varepsilon; \mathcal{K}))$

Entropy Numbers.

$e_n(\mathcal{K}) = \inf\{\varepsilon \mid H(\varepsilon; \mathcal{K}) \leq n\}$

Metric Entropy and Entropy Numbers



Covering Number.

$$\begin{aligned} N(\varepsilon;\mathcal{K}) &= \text{number of} \\ &\quad \text{to cover } \mathcal{K} \\ &= 1 \end{aligned}$$

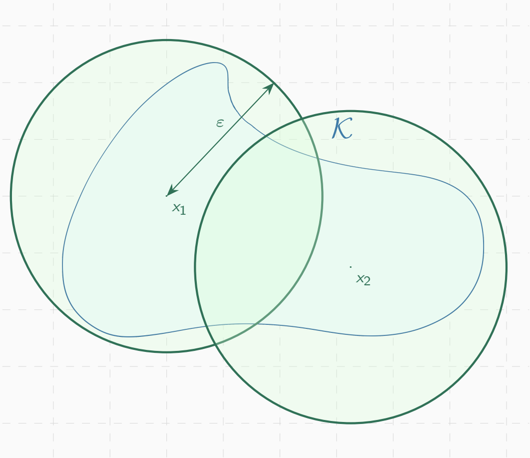
Metric Entropy.

$$\begin{aligned} H(\varepsilon;\mathcal{K}) &= \log(N(\varepsilon;\mathcal{K})) \\ &= 0 \end{aligned}$$

Entropy Numbers.

$$e_n(\mathcal{K}) = \inf\{\varepsilon \mid H(\varepsilon;\mathcal{K}) \leq n\}$$

Metric Entropy and Entropy Numbers



Covering Number.

$$N(\varepsilon; \mathcal{K}) = \text{number of } \bigcirc \text{ to cover } \mathcal{K} = 2$$

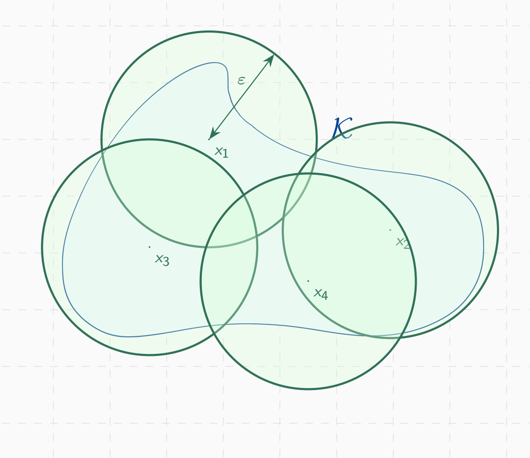
Metric Entropy.

$$H(\varepsilon; \mathcal{K}) = \log(N(\varepsilon; \mathcal{K})) = 1$$

Entropy Numbers.

$$e_n(\mathcal{K}) = \inf\{\varepsilon \mid H(\varepsilon; \mathcal{K}) \leq n\}$$

Metric Entropy and Entropy Numbers



Covering Number.

$N(\varepsilon;\mathcal{K})$ = number of  to cover $\mathcal{K}$

$= 4$

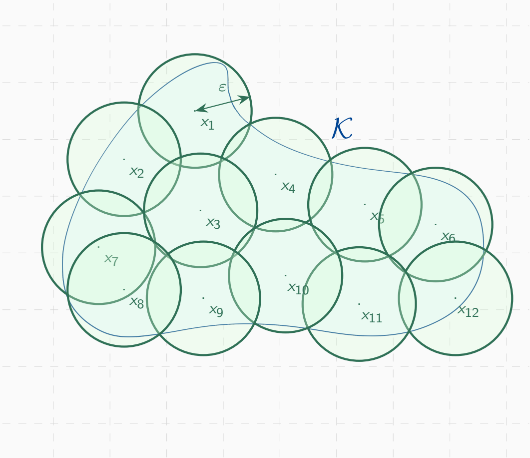
Metric Entropy.

$$H(\varepsilon;\mathcal{K}) = \log(N(\varepsilon;\mathcal{K})) = 2$$

Entropy Numbers.

$$e_n(\mathcal{K}) = \inf\{\varepsilon \mid H(\varepsilon;\mathcal{K}) \leq n\}$$

Metric Entropy and Entropy Numbers



Covering Number.

$$\begin{aligned} N(\varepsilon;\mathcal{K}) &= \text{number of} \\ &\quad \text{to cover } \mathcal{K} \\ &= 12 \end{aligned}$$

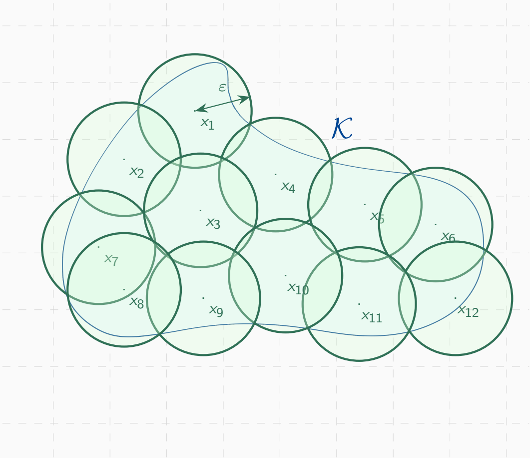
Metric Entropy.

$$\begin{aligned} H(\varepsilon;\mathcal{K}) &= \log(N(\varepsilon;\mathcal{K})) \\ &= 2 + \log(3) \end{aligned}$$

Entropy Numbers.

$$e_n(\mathcal{K}) = \inf\{\varepsilon \mid H(\varepsilon;\mathcal{K}) \le n\}$$

Metric Entropy and Entropy Numbers



Covering Number.

$$N(\varepsilon; \mathcal{K}) = \text{number of} \quad \text{to cover } \mathcal{K} = 12$$

Metric Entropy.

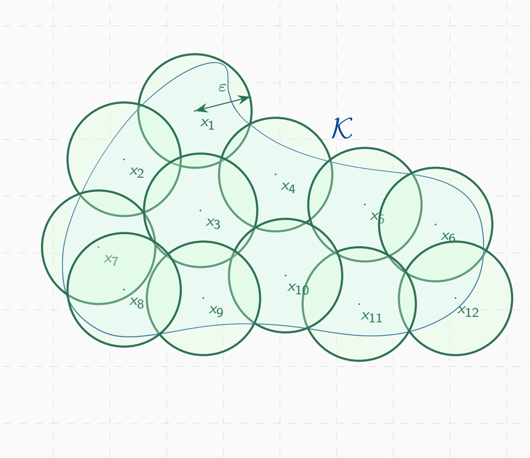
$$H(\varepsilon; \mathcal{K}) = \log(N(\varepsilon; \mathcal{K})) = 2 + \log(3)$$

Entropy Numbers.

$$e_n(\mathcal{K}) = \inf\{\varepsilon \mid H(\varepsilon; \mathcal{K}) \leq n\}$$

- Approximation Theory
- Harmonic Analysis
- Statistics

Metric Entropy and Entropy Numbers



Covering Number.

$N(\varepsilon; \mathcal{K})$ = number of 

to cover $\mathcal{K}$
$= 12$

Metric Entropy.

$$H(\varepsilon; \mathcal{K}) = \log(N(\varepsilon; \mathcal{K})) = 2 + \log(3)$$

Entropy Numbers.

$$e_n(\mathcal{K}) = \inf\{\varepsilon \mid H(\varepsilon; \mathcal{K}) \leq n\}$$

- Approximation Theory
- Harmonic Analysis
- Statistics

- Empirical Processes
- Statistical Learning Theory
- PDEs
- High-Dimensional Probability and Statistics
- Information Theory
- Linear Operator Theory
- Neural Network Theory
- Signal Processing
- and many others

Entropy of Compact Operators

Entropy and entropy numbers of $T\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

$H_T(\varepsilon) := H(\varepsilon; T(\mathcal{B})) = \log(\text{number of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ required to cover } T(\mathcal{B}))$

$e_n(T) = \inf\{\varepsilon \mid H_T(\varepsilon) \le n\}$

Entropy of Compact Operators

Entropy and entropy numbers of $T\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

$H_T(\varepsilon) := H(\varepsilon; T(\mathcal{B})) = \log(\text{number of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ required to cover } T(\mathcal{B}))$

$e_n(T) = \inf\{\varepsilon \mid H_T(\varepsilon) \leq n\}$

Standard methods include

- spectral methods;
- explicit construction of packings and coverings;
- widths and s-numbers;
- interpolation and factorization;

Entropy of Compact Operators

Entropy and entropy numbers of $T\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

$H_T(\varepsilon) := H(\varepsilon; T(\mathcal{B})) = \log(\text{number of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ required to cover } T(\mathcal{B}))$

$e_n(T) = \inf\{\varepsilon \mid H_T(\varepsilon) \leq n\}$

Standard methods include

- spectral methods;
- explicit construction of packings and coverings;
- widths and s-numbers;
- interpolation and factorization;

GOAL OF THE TALK

Leverage time-frequency information on T to easily obtain sharp characterizations of its entropy and entropy numbers!

Entropy of Compact Operators

Entropy and entropy numbers of $T\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

$H_T(\varepsilon) := H(\varepsilon; T(\mathcal{B})) = \log(\text{number of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ required to cover } T(\mathcal{B}))$

$e_n(T) = \inf\{\varepsilon \mid H_T(\varepsilon) \leq n\}$

Standard methods include

- spectral methods;
- explicit construction of packings and coverings;
- widths and s-numbers;
- interpolation and factorization;

GOAL OF THE TALK

Leverage **time-frequency information** on T to easily obtain sharp characterizations of its entropy and entropy numbers!

Entropy of Compact Operators

Entropy and entropy numbers of $T\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

$H_T(\varepsilon) := H(\varepsilon; T(\mathcal{B})) = \log(\text{number of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ required to cover } T(\mathcal{B}))$

$e_n(T) = \inf\{\varepsilon \mid H_T(\varepsilon) \leq n\}$

Standard methods include

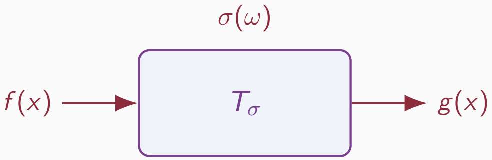
- spectral methods;
- explicit construction of packings and coverings;
- widths and s-numbers;
- interpolation and factorization;

GOAL OF THE TALK

Leverage **time-frequency information** on T to **easily** obtain **sharp** characterizations of its entropy and entropy numbers!

Time-Frequency Representation

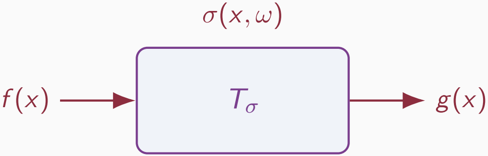
$\sigma(\omega)$ transfer function of LTI system



$$\hat{g}(\omega) = \sigma(\omega)\,\hat{f}(\omega)$$

Time-Frequency Representation

$\sigma(\omega)$ transfer function of LTI system $\;\rightarrow\;$ $\sigma(x,\omega)$ transfer function of LTV system



$$\hat{g}(\omega) = \sigma(x,\omega)\,\hat{f}(\omega)$$

Time-Frequency Representation

$\sigma(\omega)$ transfer function of LTI system $\;\rightarrow\;$ $\sigma(x,\omega)$ transfer function of LTV system

$$\sigma(x,\omega)$$

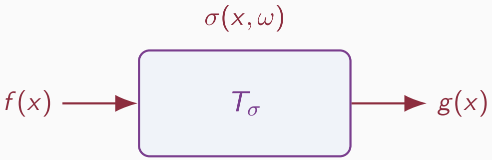
$$f(x) \longrightarrow \boxed{\;T_\sigma\;} \longrightarrow g(x)$$

$$\hat{g}(\omega) = \sigma(x,\omega)\,\hat{f}(\omega)$$

<u>Pseudodifferential Operators</u>: $\quad T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\,\hat{f}(\omega)\,e^{2\pi i\,x\cdot\omega}\,d\omega$

Time-Frequency Representation

$\sigma(\omega)$ transfer function of LTI system $\rightarrow$ $\sigma(x,\omega)$ transfer function of LTV system

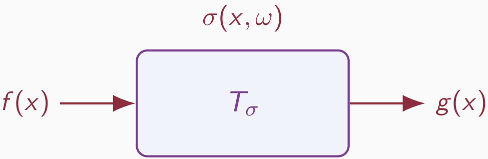


- $\sigma(x,\omega) = 1$: $T_\sigma = \mathrm{Id}$

Pseudodifferential Operators: $$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\,\hat{f}(\omega)\,e^{2\pi i\,x\cdot\omega}\,d\omega$$

Time-Frequency Representation

$\sigma(\omega)$ transfer function of LTI system $\rightarrow$ $\sigma(x, \omega)$ transfer function of LTV system

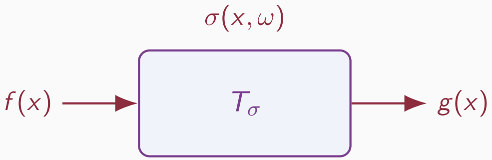


- $\sigma(x, \omega) = 1$: $T_\sigma = \mathrm{Id}$
- $\sigma(x, \omega) = u(x)$: $T_\sigma f(x) = u(x) f(x)$

Pseudodifferential Operators: $$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x, \omega)\, \hat{f}(\omega)\, e^{2\pi i\, x \cdot \omega}\, d\omega$$

Time-Frequency Representation

$\sigma(\omega)$ transfer function of LTI system $\rightarrow$ $\sigma(x,\omega)$ transfer function of LTV system

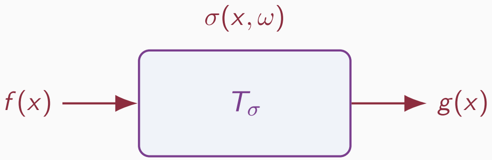


- $\sigma(x,\omega) = 1$: $T_\sigma = \mathrm{Id}$
- $\sigma(x,\omega) = u(x)$: $T_\sigma f(x) = u(x) f(x)$
- $\sigma(x,\omega) = v(\omega)$: LTI System

Pseudodifferential Operators: $$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\, \hat{f}(\omega)\, e^{2\pi i\, x\cdot\omega}\, d\omega$$

Time-Frequency Representation

$\sigma(\omega)$ transfer function of LTI system $\rightarrow$ $\sigma(x,\omega)$ transfer function of LTV system



- $\sigma(x,\omega) = 1$: $T_\sigma = \mathrm{Id}$
- $\sigma(x,\omega) = u(x)$: $T_\sigma f(x) = u(x) f(x)$
- $\sigma(x,\omega) = v(\omega)$: LTI System

Pseudodifferential Operators: $$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\, \hat{f}(\omega)\, e^{2\pi i\, x\cdot\omega}\, d\omega$$

Differential Operators are Pseudodifferential Operators with polynomial symbol.

The **Laplace Operator** $-\Delta = -\partial_1^2 - \cdots - \partial_d^2$ has symbol $\sigma(x,\omega) = (2\pi\|\omega\|)^2$.

Main Result

Time-frequency Representation

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\, \hat{f}(\omega)\, e^{2\pi i x\cdot\omega}\, d\omega$$

Entropy

$$H_\sigma(\varepsilon) = \log(\text{nb of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ to cover } T_\sigma(\mathcal{B}))$$

Question: How does the entropy of T_σ depend on σ?

Main Result

Time-frequency Representation

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\, \hat{f}(\omega)\, e^{2\pi i x\cdot\omega}\, d\omega$$

Entropy

$$H_\sigma(\varepsilon) = \log(\text{nb of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ to cover } T_\sigma(\mathcal{B}))$$

Question: How does the entropy of T_σ depend on σ?

Theorem — A. & Bölcskei '26

If $\sigma > 0$ is a negative-order hypoelliptic symbol with regular variations, then

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Main Result

Time-frequency Representation

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\, \hat{f}(\omega)\, e^{2\pi i x\cdot\omega}\, d\omega$$

Entropy

$$H_\sigma(\varepsilon) = \log(\text{nb of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ to cover } T_\sigma(\mathcal{B}))$$

Question: How does the entropy of T_σ depend on σ?

Theorem — A. & Bölcskei '26

If $\boxed{\sigma > 0}$ is a negative-order hypoelliptic symbol with regular variations, then

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Technical assumption, can easily be weakened;

Main Result

Time-frequency Representation

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\, \hat{f}(\omega)\, e^{2\pi i x\cdot\omega}\, d\omega$$

Entropy

$$H_\sigma(\varepsilon) = \log(\text{nb of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ to cover } T_\sigma(\mathcal{B}))$$

Question: How does the entropy of T_σ depend on σ?

Theorem — A. & Bölcskei '26

If $\boxed{\sigma > 0}$ is a negative-order hypoelliptic symbol with $\boxed{\text{regular variations}}$, then

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\!\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Technical assumption, can easily be weakened;

Ensures no large oscillations at infinity;

Main Result

Time-frequency Representation

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\, \hat{f}(\omega)\, e^{2\pi i x\cdot\omega}\, d\omega$$

Entropy

$$H_\sigma(\varepsilon) = \log(\text{nb of } L^2\text{-balls } \varepsilon\mathcal{B} \text{ to cover } T_\sigma(\mathcal{B}))$$

Question: How does the entropy of T_σ depend on σ?

Theorem — A. & Bölcskei '26

If $\boxed{\sigma > 0}$ is a $\boxed{\text{negative-order hypoelliptic}}$ symbol with $\boxed{\text{regular variations}}$, then

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Technical assumption, can easily be weakened;

Ensures no large oscillations at infinity;

$$C_1\left(1+\|z\|^2\right)^{-m_1} \le |\sigma(z)| \le C_2\left(1+\|z\|^2\right)^{-m_2}$$

$$\frac{|\partial^\alpha \sigma(z)|}{|\sigma(z)|} \le C_\alpha\left(1+\|z\|^2\right)^{-\rho|\alpha|}$$

In Practice

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

In Practice

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Theorem — A. & Bölcskei '26

If $\sigma > 0$ is a negative-order hypoelliptic symbol with regular variations, then

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \operatorname{vol}\left\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\right\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

In Practice

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\!\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Theorem — A. & Bölcskei '26

If $\sigma > 0$ is a negative-order hypoelliptic symbol with regular variations, then

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\left\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\right\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

Simple **plug-and-play formula**: no need for explicit covering and packing constructions

In Practice

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Theorem — A. & Bölcskei '26

If $\sigma > 0$ is a negative-order hypoelliptic symbol with regular variations, then

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

Simple **plug-and-play formula**: no need for explicit covering and packing constructions

$\sigma(x,\omega) \sim (\|x\|^2 + \|\omega\|^2)^{-\frac{k}{2}}$

In Practice

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Theorem — A. & Bölcskei '26

If $\sigma > 0$ is a negative-order hypoelliptic symbol with regular variations, then

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\left\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\right\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

Simple **plug-and-play formula**: no need for explicit covering and packing constructions

$$\sigma(x,\omega) \sim (\|x\|^2 + \|\omega\|^2)^{-\frac{k}{2}} \implies \mathrm{vol}\{\sigma(x,\omega) > \lambda\} \sim \nu_{2d}\, \lambda^{-\frac{2d}{k}}$$

In Practice

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Theorem — A. & Bölcskei '26

If $\sigma > 0$ is a negative-order hypoelliptic symbol with regular variations, then

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\left\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\right\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

Simple **plug-and-play formula**: no need for explicit covering and packing constructions

$$\sigma(x,\omega) \sim (\|x\|^2 + \|\omega\|^2)^{-\frac{k}{2}} \implies \mathrm{vol}\{\sigma(x,\omega) > \lambda\} \sim \nu_{2d}\, \lambda^{-\frac{2d}{k}}$$

$$\implies H_\sigma(\varepsilon) \sim \frac{k\, \nu_{2d}}{2d}\, \varepsilon^{-\frac{2d}{k}}$$

In Practice

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Theorem — A. & Bölcskei '26

If $\sigma > 0$ is a negative-order hypoelliptic symbol with regular variations, then

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\left\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\right\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

Simple **plug-and-play formula**: no need for explicit covering and packing constructions

$$\sigma(x,\omega) \sim (\|x\|^2 + \|\omega\|^2)^{-\frac{k}{2}} \implies \mathrm{vol}\{\sigma(x,\omega) > \lambda\} \sim \nu_{2d}\, \lambda^{-\frac{2d}{k}}$$

$$\implies H_\sigma(\varepsilon) \sim \frac{k\,\nu_{2d}}{2d}\, \varepsilon^{-\frac{2d}{k}} \quad \text{and} \quad e_n(T_\sigma) \sim \left(\frac{k\,\nu_{2d}}{2d\, n}\right)^{\frac{2d}{k}}$$

In Practice

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega \qquad (\varepsilon \to 0)$$

Theorem — A. & Bölcskei '26

If $\sigma > 0$ is a negative-order hypoelliptic symbol with regular variations, then

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \operatorname{vol}\left\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\right\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

Simple **plug-and-play formula**: no need for explicit covering and packing constructions

$$\sigma(x,\omega) \sim (\|x\|^2 + \|\omega\|^2)^{-\frac{k}{2}} \implies \operatorname{vol}\{\sigma(x,\omega) > \lambda\} \sim \nu_{2d}\, \lambda^{-\frac{2d}{k}}$$

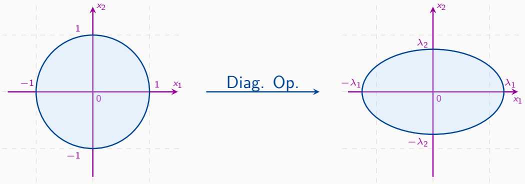
$$\implies H_\sigma(\varepsilon) \sim \frac{k\, \nu_{2d}}{2d}\, \varepsilon^{-\frac{2d}{k}} \quad \text{and} \quad e_n(T_\sigma) \sim \left(\frac{k\, \nu_{2d}}{2d\, n}\right)^{\frac{2d}{k}}$$

Provides **sharp asymptotics** (cf. applications).

Proof Sketch

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

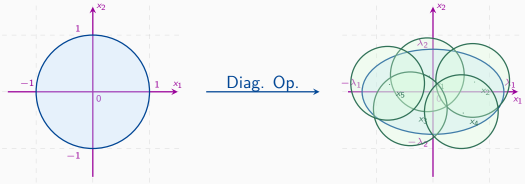
1/ Metric Entropy of Ellipsoids.



Proof Sketch

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

1/ Metric Entropy of Ellipsoids.



Effective Dimension: $\lambda_d \simeq \varepsilon$

Lower Bound: Volume Arguments

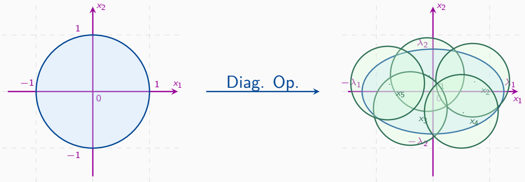
Upper Bound: Block Decomposition
$+$ Rogers' Bound

$$\varepsilon^d\, N(\varepsilon)\, \mathrm{vol}(\bigcirc) \simeq \mathrm{vol}(\;) = \left(\prod_{n=1}^{d} \lambda_n\right) \mathrm{vol}(\bigcirc)$$

Proof Sketch

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega)\in\mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\}\,\frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

1/ Metric Entropy of Ellipsoids.



<u>Effective Dimension</u>: $\lambda_d \simeq \varepsilon$

<u>Lower Bound</u>: Volume Arguments

<u>Upper Bound</u>: Block Decomposition
$+$ Rogers' Bound

$$H_\sigma(\varepsilon) \sim \sum_n \log_+\left(\frac{\lambda_n}{\varepsilon}\right) = \int_\varepsilon^\infty M(\lambda)\,\frac{d\lambda}{\lambda} \quad \text{with} \quad M(\lambda) = \#\{\lambda_n > \lambda\}$$

Proof Sketch

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) \; > \; \lambda\Big\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

1/ Metric Entropy of Ellipsoids. $H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty M(\lambda) \frac{d\lambda}{\lambda}$ with $M(\lambda) = \#\{\lambda_n > \lambda\}$

Proof Sketch

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

1/ Metric Entropy of Ellipsoids. $H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty M(\lambda) \frac{d\lambda}{\lambda}$ with $M(\lambda) = \#\{\lambda_n > \lambda\}$

2/ Microlocal Analysis.

Generalized Weyl Formula: $M(\lambda) \sim \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\} \qquad (\lambda \to 0)$

- $\{\psi_n\}_n$ orthonormal basis of eigenfunctions in $L^2(\mathbb{R}^d)$;
- by the **Uncertainty Principle**, ψ_n is localized in unit-volume regions of phase space centered around (x_n, ω_n);
- the supports of $\{\psi_n\}_n$ are essentially disjoint in phase space;
- ψ_n's eigenvalue $\lambda_n \simeq \sigma(x_n, \omega_n)$ $\to$ keep only $\sigma(x_n, \omega_n) > \lambda$.

Proof Sketch

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

1/ Metric Entropy of Ellipsoids. $H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty M(\lambda)\,\frac{d\lambda}{\lambda}$ with $M(\lambda) = \#\{\lambda_n > \lambda\}$

2/ Microlocal Analysis. $M(\lambda) \sim \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\}$

Proof Sketch

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\} \frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

1/ Metric Entropy of Ellipsoids. $H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty M(\lambda) \frac{d\lambda}{\lambda}$ with $M(\lambda) = \#\{\lambda_n > \lambda\}$

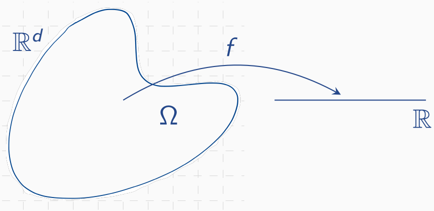
2/ Microlocal Analysis. $M(\lambda) \sim \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\}$

3/ Combine and Apply Fubini.

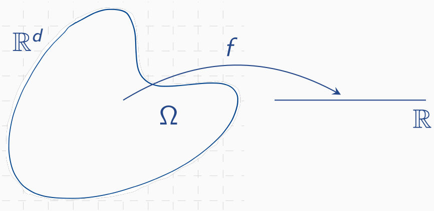
$$\begin{aligned} H_\sigma(\varepsilon) &\sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\} \frac{d\lambda}{\lambda} \\ &= \int_\varepsilon^\infty \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{\{\sigma(x,\omega) > \lambda\}}}{\lambda}\, dx\, d\omega\, d\lambda = \int_{\mathbb{R}^{2d}} \log_+\!\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega. \end{aligned}$$

Applications

Application to Sobolev Spaces — Bounded Domains

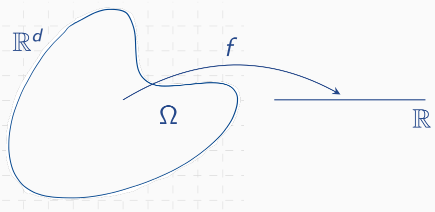


Application to Sobolev Spaces — Bounded Domains



- f is square-integrable
- the k-th derivatives of f are square-integrable
- f vanishes on the boundary $\partial\Omega$

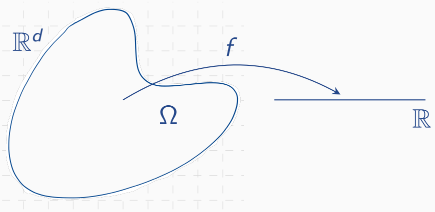
Application to Sobolev Spaces — Bounded Domains



- f is square-integrable
- the k-th derivatives of f are square-integrable
- f vanishes on the boundary $\partial\Omega$

$$\mathrm{Sob}^{\Omega}_{k,d} = T\big(L^2(\Omega)\big), \quad T = [\mathrm{Id} - \Delta]^{-\frac{k}{2}}$$

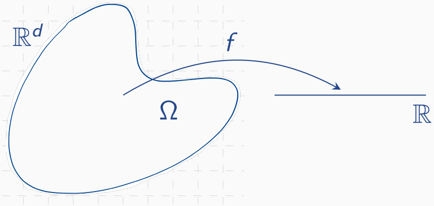
Application to Sobolev Spaces — Bounded Domains



- f is square-integrable
- the k-th derivatives of f are square-integrable
- f vanishes on the boundary $\partial\Omega$

$$\mathrm{Sob}_{k,d}^{\Omega} = T\big(L^2(\Omega)\big), \quad T = [\mathrm{Id} - \Delta]^{-\frac{k}{2}} \quad \text{with} \;\; \sigma(x,\omega) \sim \big(1 + (2\pi\|\omega\|)^2\big)^{-\frac{k}{2}}$$

Application to Sobolev Spaces — Bounded Domains



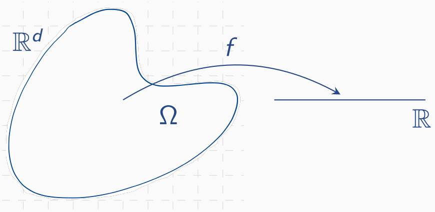
- f is square-integrable
- the k-th derivatives of f are square-integrable
- f vanishes on the boundary $\partial\Omega$

$$\mathrm{Sob}_{k,d}^{\Omega} = T\big(L^2(\Omega)\big), \quad T = [\mathrm{Id} - \Delta]^{-\frac{k}{2}} \quad \text{with} \quad \sigma(x,\omega) \sim \big(1 + (2\pi\|\omega\|)^2\big)^{-\frac{k}{2}}$$

The unit ball $\mathcal{B}$ in $\mathrm{Sob}_{k,d}^{\Omega}$ has metric entropy

$$H(\varepsilon;\mathcal{B}) \sim \int_{\varepsilon}^{\infty} \mathrm{vol}\{(x,\omega) \in \Omega \times \mathbb{R}^d \mid \sigma(x,\omega) > \lambda\} \frac{d\lambda}{\lambda}$$

Application to Sobolev Spaces — Bounded Domains



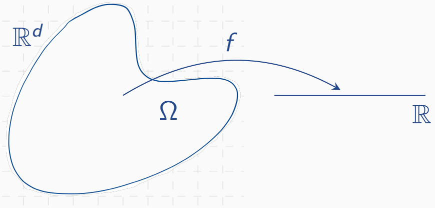
- f is square-integrable
- the k-th derivatives of f are square-integrable
- f vanishes on the boundary $\partial\Omega$

$$\mathrm{Sob}^{\Omega}_{k,d} = T\big(L^2(\Omega)\big), \quad T = [\mathrm{Id} - \Delta]^{-\frac{k}{2}} \quad \text{with} \quad \sigma(x,\omega) \sim \big(1 + (2\pi\|\omega\|)^2\big)^{-\frac{k}{2}}$$

The unit ball $\mathcal{B}$ in $\mathrm{Sob}^{\Omega}_{k,d}$ has metric entropy

$$H(\varepsilon;\mathcal{B}) \sim \int_{\varepsilon}^{\infty} \mathrm{vol}\big\{(x,\omega) \in \Omega \times \mathbb{R}^d \mid \sigma(x,\omega) > \lambda\big\} \frac{d\lambda}{\lambda} \sim \frac{k\,\nu_d\,\mathrm{vol}(\Omega)}{d\,(2\pi)^d}\,\varepsilon^{-\frac{d}{k}}$$

Application to Sobolev Spaces — Bounded Domains



- f is square-integrable
- the k-th derivatives of f are square-integrable
- f vanishes on the boundary $\partial\Omega$

$$\mathrm{Sob}^{\Omega}_{k,d} = T\big(L^2(\Omega)\big), \quad T = [\mathrm{Id} - \Delta]^{-\frac{k}{2}} \quad \text{with} \quad \sigma(x,\omega) \sim \big(1 + (2\pi\|\omega\|)^2\big)^{-\frac{k}{2}}$$

The unit ball $\mathcal{B}$ in $\mathrm{Sob}^{\Omega}_{k,d}$ has metric entropy

$$H(\varepsilon;\mathcal{B}) \sim \int_{\varepsilon}^{\infty} \mathrm{vol}\{(x,\omega) \in \Omega \times \mathbb{R}^d \mid \sigma(x,\omega) > \lambda\} \frac{d\lambda}{\lambda} \sim \frac{k\,\nu_d\,\mathrm{vol}(\Omega)}{d\,(2\pi)^d}\,\varepsilon^{-\frac{d}{k}}$$

Application to Sobolev Spaces — Unbounded Domains

Sobolev balls are **not compact** in $L^2(\mathbb{R}^d)$ for unbounded domains!

No canonical way to generalize to unbounded domains, we consider two:

Sobolev Spaces with Potential

$$\mathrm{Sob}_{k,d}^{\text{pot.}} = T\big(L^2(\mathbb{R}^d)\big), \quad T = [\mathrm{Id} + U - \Delta]^{-\frac{k}{2}}, \quad \sigma(x,\omega) \sim \big(1 + u(x) + (2\pi\|\omega\|)^2\big)^{-\frac{k}{2}}$$

Application to Sobolev Spaces — Unbounded Domains

Sobolev balls are **not compact** in $L^2(\mathbb{R}^d)$ for unbounded domains!

No canonical way to generalize to unbounded domains, we consider two:

Sobolev Spaces with Potential

$$\mathrm{Sob}_{k,d}^{\text{pot.}} = T\big(L^2(\mathbb{R}^d)\big), \quad T = [\mathrm{Id} + U - \Delta]^{-\frac{k}{2}}, \quad \sigma(x,\omega) \sim \big(1 + u(x) + (2\pi\|\omega\|)^2\big)^{-\frac{k}{2}}$$

Theorem — A. & Bölcskei '26

The unit ball $\mathcal{B}$ in $\mathrm{Sob}_{k,d}^{\text{pot.}}$ with $u(x) = c\|x\|^2$ has metric entropy

$$H(\varepsilon;\mathcal{B}) \sim \int_{\varepsilon}^{\infty} \mathrm{vol}\big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\big\} \frac{d\lambda}{\lambda} \sim \frac{k\,\nu_{2d}}{2d\,(2\pi\sqrt{c})^d}\,\varepsilon^{-\frac{2d}{k}}$$

Application to Sobolev Spaces — Unbounded Domains

Sobolev Spaces with Potential

$$\sigma(x,\omega) = \left(1 + u(x) + (2\pi\|\omega\|)^2\right)^{-\frac{k}{2}} \quad \text{and} \quad H(\varepsilon;\mathcal{B}) \sim \frac{k\,\nu_{2d}}{2d\,(2\pi\sqrt{c})^d}\,\varepsilon^{-\frac{2d}{k}}$$

Application to Sobolev Spaces — Unbounded Domains

Sobolev Spaces with Potential

$$\sigma(x,\omega) = \left(1 + u(x) + (2\pi\|\omega\|)^2\right)^{-\frac{k}{2}} \quad \text{and} \quad H(\varepsilon;\mathcal{B}) \sim \frac{k\,\nu_{2d}}{2d\,(2\pi\sqrt{c})^d}\,\varepsilon^{-\frac{2d}{k}}$$

Weighted Sobolev Spaces

$$\mathrm{Sob}^{\mathrm{w}}_{k,d} = T\left(L^2(\mathbb{R}^d)\right),\ \ T = [\mathrm{Id} - \Delta]^{-\frac{k}{2}}[\mathrm{Id} + U]^{-\frac{r}{2}},\ \sigma(x,\omega) \sim \left(1 + (2\pi\|\omega\|)^2\right)^{-\frac{k}{2}}(1 + u(x))^{-\frac{r}{2}}$$

Application to Sobolev Spaces — Unbounded Domains

Sobolev Spaces with Potential

$$\sigma(x,\omega) = \left(1 + u(x) + (2\pi\|\omega\|)^2\right)^{-\frac{k}{2}} \quad \text{and} \quad H(\varepsilon;\mathcal{B}) \sim \frac{k\,\nu_{2d}}{2d\,(2\pi\sqrt{c})^d}\,\varepsilon^{-\frac{2d}{k}}$$

Weighted Sobolev Spaces

$$\mathrm{Sob}^{\mathrm{w}}_{k,d} = T\left(L^2(\mathbb{R}^d)\right),\ T = [\mathrm{Id} - \Delta]^{-\frac{k}{2}}[\mathrm{Id} + U]^{-\frac{r}{2}},\ \sigma(x,\omega) \sim \left(1 + (2\pi\|\omega\|)^2\right)^{-\frac{k}{2}}(1 + u(x))^{-\frac{r}{2}}$$

Theorem — A. & Bölcskei '26

The unit ball $\mathcal{B}$ in $\mathrm{Sob}^{\mathrm{w}}_{k,d}$ with $u(x) = c\|x\|^2$ has metric entropy

$$H(\varepsilon;\mathcal{B}) \sim \int_{\varepsilon}^{\infty} \mathrm{vol}\left\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\right\} \frac{d\lambda}{\lambda} \sim \frac{\nu_d^2}{(2\pi\sqrt{c})^d} \begin{cases} C_{r,k,d}\,\varepsilon^{-\frac{d}{\min\{r,k\}}} & \text{if } r \neq k \\ \varepsilon^{-\frac{d}{k}} \ln\left(\varepsilon^{-1}\right) & \text{if } r = k \end{cases}$$

Application to Nonparametric Statistics

Estimate x from noisy observations $y = x + \kappa\,\xi$

Application to Nonparametric Statistics

Estimate x from noisy observations $y = x + \kappa\xi$

$$x \in T_\sigma(\mathcal{B}), \qquad \kappa > 0, \qquad \xi_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$$

Application to Nonparametric Statistics

Estimate x from noisy observations $y = x + \kappa\,\xi$

$x \in T_\sigma(\mathcal{B})$, $\qquad \kappa > 0$, $\qquad \xi_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$

Risk: $\quad R_\sigma(\kappa) := \inf_{\hat{x}_\kappa} \sup_x \mathbb{E}_{y\sim x}\Big[\big\|\hat{x}_\kappa(y) - x\big\|^2\Big]$

Application to Nonparametric Statistics

Estimate x from noisy observations $y = x + \kappa\,\xi$

$x \in T_\sigma(\mathcal{B}), \qquad \kappa > 0, \qquad \xi_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$

Risk: $\quad R_\sigma(\kappa) := \inf_{\hat{x}_\kappa} \sup_x \mathbb{E}_{y\sim x}\left[\left\|\hat{x}_\kappa(y) - x\right\|^2\right]$

- Density Estimation
- Spectral Density Estimation
- Nonparamteric Regression

Application to Nonparametric Statistics

Estimate x from noisy observations $y = x + \kappa\,\xi$

$x \in T_\sigma(\mathcal{B}), \qquad \kappa > 0, \qquad \xi_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$

Risk: $\quad R_\sigma(\kappa) := \inf_{\hat{x}_\kappa} \sup_x \mathbb{E}_{y\sim x}\Big[\big\|\hat{x}_\kappa(y) - x\big\|^2\Big]$

- Density Estimation
- Spectral Density Estimation
- Nonparamteric Regression

Theorem — A. & Bölcskei '26

Under the usual assumptions

$$R_\sigma(\kappa) \sim \kappa^2 \int_{\mathbb{R}^{2d}} \left(1 - \frac{\varepsilon_\kappa}{\sigma(x,\omega)}\right)_+ dx\; d\omega \qquad (\kappa \to 0)$$

where ε_κ is determined (implicitly) by

$$\kappa^2 \int_{\mathbb{R}^{2d}} \frac{1}{\sigma(x,\omega)\,\varepsilon_\kappa}\left(1 - \frac{\varepsilon_\kappa}{\sigma(x,\omega)}\right)_+ dx\; d\omega = 1 \qquad (\kappa > 0)$$

Application to Nonparametric Statistics — Pinsker's Theorem

Only few classes for which asymptotics of minimax risk are precisely known.

Pinsker's Theorem for Sobolev spaces on $[0, 1]$. Our results generalize it to:

Application to Nonparametric Statistics — Pinsker's Theorem

Only few classes for which asymptotics of minimax risk are precisely known.

Pinsker's Theorem for Sobolev spaces on $[0, 1]$. Our results generalize it to:

Sobolev Spaces on Bounded Domain $\Omega \subset \mathbb{R}^d$.

$$R(\kappa) \sim \frac{d+2k}{d} \left(\frac{k\, d\, \nu_d \operatorname{vol}(\Omega)\, \kappa^2}{(2\pi)^d (d+k)(d+2k)} \right)^{\frac{2k}{d+2k}} \qquad (\kappa \to 0)$$

Application to Nonparametric Statistics — Pinsker's Theorem

Only few classes for which asymptotics of minimax risk are precisely known.

Pinsker's Theorem for Sobolev spaces on $[0,1]$. Our results generalize it to:

Sobolev Spaces on Bounded Domain $\Omega \subset \mathbb{R}^d$.

$$R(\kappa) \sim \frac{d+2k}{d}\left(\frac{k\, d\, \nu_d \operatorname{vol}(\Omega)\, \kappa^2}{(2\pi)^d (d+k)(d+2k)}\right)^{\frac{2k}{d+2k}} \qquad (\kappa \to 0)$$

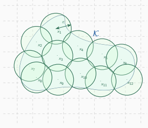
Sobolev Spaces on $\mathbb{R}^d$ with Quadratic Potential.

$$R(\kappa) \sim \frac{d+k}{d}\left(\frac{d\, k\, \nu_{2d}\, \kappa^2}{(2\pi\sqrt{c})^d (d+k)(2d+k)}\right)^{\frac{k}{d+k}} \qquad (\kappa \to 0)$$

Thank You

Summary

Metric Entropy and Entropy Numbers



Covering Number.

$N(\varepsilon;\mathcal{K})$ = number of to cover $\mathcal{K}$ $= 12$

Metric Entropy.

$$H(\varepsilon;\mathcal{K}) = \log(N(\varepsilon;\mathcal{K})) = 2 + \log(3)$$

Entropy Numbers.

$$e_n(\mathcal{K}) = \inf\{\varepsilon \mid H(\varepsilon;\mathcal{K}) \leq n\}$$

- Approximation Theory
- Harmonic Analysis
- Statistics

• Empirical Processes • Statistical Learning Theory • PDEs • High-Dimensional Probability and Statistics • Information Theory • Linear Operator Theory • Neural Network Theory • Signal Processing • and many others

Main Result

Time-frequency Representation

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\,\hat{f}(\omega)\,e^{2\pi i x\cdot\omega}\,d\omega$$

Entropy

Number of L^2-balls $\varepsilon\mathcal{B}$ to cover $T_\sigma(\mathcal{B})$

Question: How does the entropy of T_σ depend on σ?

Theorem — A. & Bölcskei '26

If $\boxed{\sigma > 0}$ is a $\boxed{\text{negative-order hypoelliptic}}$ symbol with $\boxed{\text{regular variations}}$, then

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \log_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\,d\omega \qquad (\varepsilon \to 0)$$

$\boxed{\text{Technical assumption, can easily be weakened;}}$

$\boxed{\text{Ensures no large oscillations at infinity;}}$

$$\boxed{\begin{gathered} C_1\left(1+\|z\|_2^2\right)^{-m_1} \leq |\sigma(z)| \leq C_2\left(1+\|z\|_2^2\right)^{-m_2} \\ \frac{|\partial^\alpha\sigma(z)|}{|\sigma(z)|} \leq C_\alpha\left(1+\|z\|_2^2\right)^{-\rho|\alpha|} \end{gathered}}$$

Proof Sketch

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\}\,\frac{d\lambda}{\lambda} \qquad (\varepsilon \to 0)$$

1/ Metric Entropy of Ellipsoids. $H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty M(\lambda)\,\frac{d\lambda}{\lambda}$ with $M(\lambda) = \#\{\lambda_n > \lambda\}$

2/ Microlocal Analysis. $M(\lambda) \sim \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\}$

3/ Combine and Apply Fubini.

$$\begin{aligned} H_\sigma(\varepsilon) &\sim \int_\varepsilon^\infty \mathrm{vol}\Big\{(x,\omega) \in \mathbb{R}^{2d} \mid \sigma(x,\omega) > \lambda\Big\}\,\frac{d\lambda}{\lambda} \\ &= \int_\varepsilon^\infty \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda\}}}{\lambda}\,dx\,d\omega\,d\lambda = \int_{\mathbb{R}^{2d}} \ln_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\,d\omega. \end{aligned}$$

Application to Nonparametric Statistics — Pinsker's Theorem

Only few classes for which asymptotics of minimax risk are precisely known.

Pinsker's Theorem for Sobolev spaces on $[0,1]$. Our results generalize it to:

Sobolev Spaces on Bounded Domain $\Omega \subset \mathbb{R}^d$.

$$R(\kappa) \sim \frac{d+2k}{d}\left(\frac{k\,d\,\omega_d\,\mathcal{H}^d(\Omega)\,\kappa^2}{(2\pi)^d(d+k)(d+2k)}\right)^{\frac{2k}{d+2k}} \qquad (\kappa \to 0)$$

Sobolev Spaces on $\mathbb{R}^d$ with Quadratic Potential.

$$R(\kappa) \sim \frac{d+k}{d}\left(\frac{d\,k\,\omega_{2d}\,\kappa^2}{(2\pi\sqrt{c})^d(d+k)(2d+k)}\right)^{\frac{k}{d+k}} \qquad (\kappa \to 0)$$

Link to Paper

https://arxiv.org/abs/2603.23744