

Entropy of compact operators with applications to Landau–Pollak–Slepian theory and Sobolev spaces

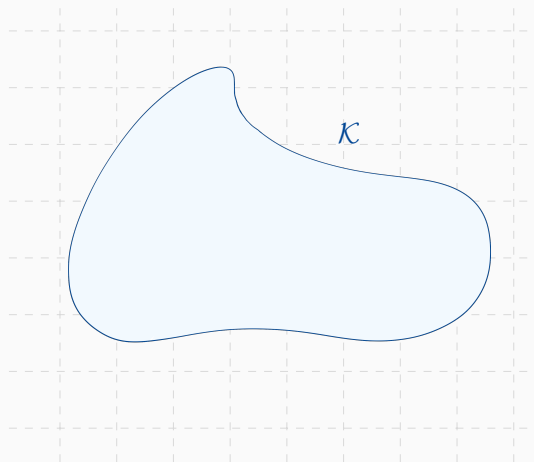
Fall Fourier Talks 2026, University of Maryland

Thomas Allard

Stanford University

October 9, 2026

Metric Entropy and Entropy Numbers



Covering Number.

$N(\varepsilon; \mathcal{K})$ = number of 
to cover $\mathcal{K}$

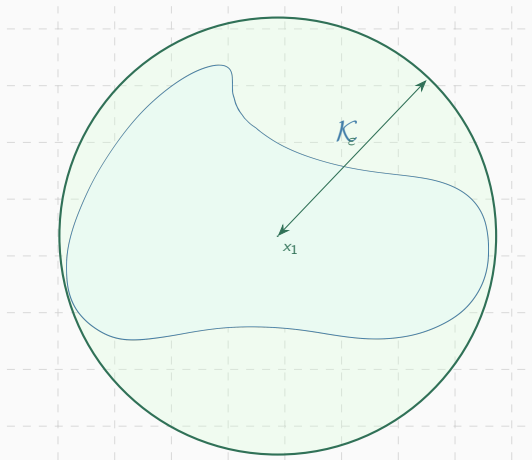
Metric Entropy.

$$H(\varepsilon; \mathcal{K}) = \log(N(\varepsilon; \mathcal{K}))$$

Entropy Numbers.

$$e_n(\mathcal{K}) = \inf\{\varepsilon \mid H(\varepsilon; \mathcal{K}) \leq n\}$$

Metric Entropy and Entropy Numbers



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$$N(\varepsilon; \mathcal{K}) = \text{number of } \begin{array}{c} \varepsilon\text{-balls} \\ \text{to cover } \mathcal{K} \end{array} = 1$$

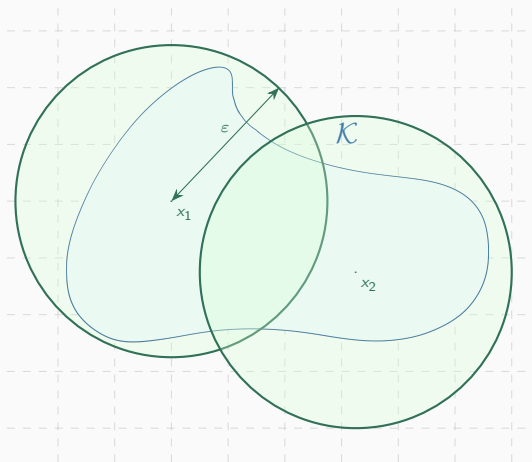
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Metric Entropy and Entropy Numbers



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$$N(\varepsilon; \mathcal{K}) = \text{number of } \textcircled{\varepsilon} \text{ to cover } \mathcal{K} \\ = 2$$

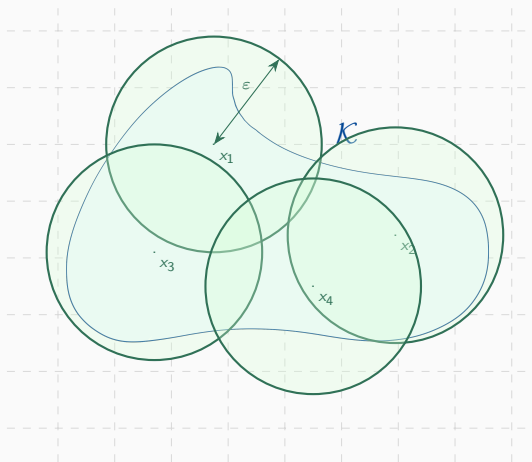
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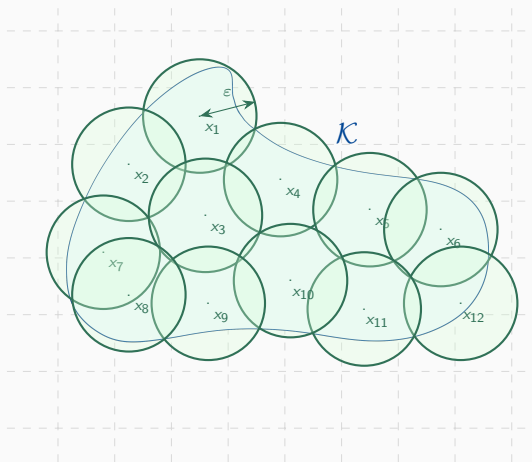
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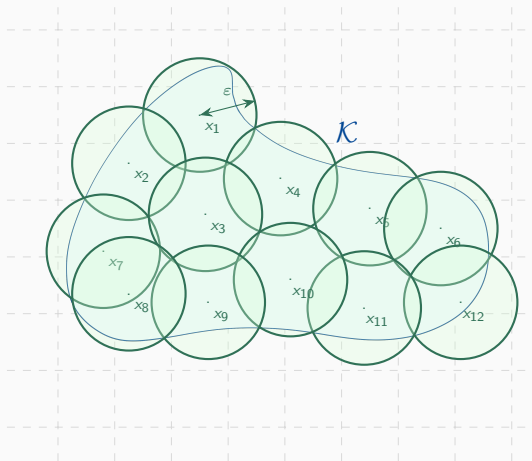
Metric Entropy.

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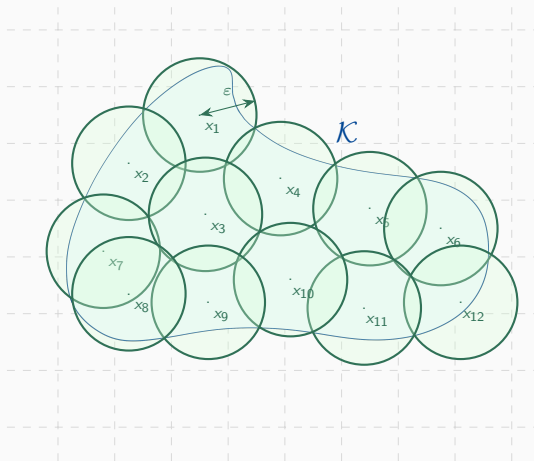
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- Statistics

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If $\lambda_n(T) = c_1 n^{-\alpha_1} + c_2 n^{-\alpha_2} + o_{n \rightarrow \infty}(n^{-\alpha_2})$ with $0 < \alpha_1 < \alpha_2 < \alpha_1 + 1/2$, then

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$$H(\varepsilon) = \alpha_1 \left(\frac{c_1}{\varepsilon} \right)^{\frac{1}{\alpha_1}} + \frac{(c_1/c_2)^{\frac{1-\alpha_2}{\alpha_1}}}{\alpha_1 - \alpha_2 + 1} \left(\frac{c_2}{\varepsilon} \right)^{\frac{1+\alpha_1-\alpha_2}{\alpha_1}} + o_{\varepsilon \rightarrow \infty} \left(\varepsilon^{-\frac{1+\alpha_1-\alpha_2}{\alpha_1}} \right),$$

$$\text{and} \quad e_m = c_1 \alpha_1^{\alpha_1} m^{-\alpha_1} + \frac{c_1 \alpha_1^{\alpha_2}}{1 + \alpha_1 - \alpha_2} m^{-\alpha_2} + o_{m \rightarrow \infty}(m^{-\alpha_2})$$

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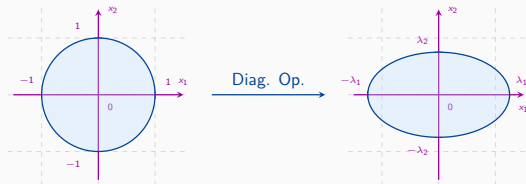
Reformulation. In terms of the **eigenvalue-counting function** $M(\lambda) = \#\{\lambda_n(T) \geq \lambda\}$

If $M(\lambda) = \kappa_1 \lambda^{-\beta_1} + \kappa_2 \lambda^{-\beta_2} + o_{\lambda \rightarrow 0}(\lambda^{-\beta_2})$, with $0 < \beta_1/2 < \beta_2 < \beta_1$, then

$$H(\varepsilon) = \frac{\kappa_1}{\beta_1} \varepsilon^{-\beta_1} + \frac{\kappa_2}{\beta_2} \varepsilon^{-\beta_2} + o_{\varepsilon \rightarrow 0}(\varepsilon^{-\beta_2})$$

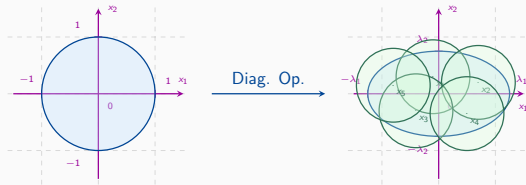
Proof Sketch

$$\lambda_n(T) \simeq \frac{c_1}{n^{\alpha_1}} + \frac{c_2}{n^{\alpha_2}} \quad \text{implies} \quad H(\varepsilon) \simeq \alpha_1 \left(\frac{c_1}{\varepsilon} \right)^{\frac{1}{\alpha_1}} + \frac{(c_1/c_2)^{\frac{1-\alpha_2}{\alpha_1}}}{\alpha_1 - \alpha_2 + 1} \left(\frac{c_2}{\varepsilon} \right)^{\frac{1+\alpha_1-\alpha_2}{\alpha_1}}$$



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Effective Dimension: $\lambda_d \simeq \varepsilon$

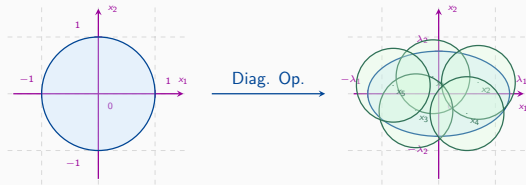
Lower Bound: Volume Arguments

Upper Bound: Block Decomposition
+ Rogers' Bound

$$\varepsilon^d N(\varepsilon) \text{vol}(\text{green circle}) \simeq \text{vol}(\text{blue oval}) = \left(\prod_{n=1}^d \lambda_n \right) \text{vol}(\text{green circle})$$

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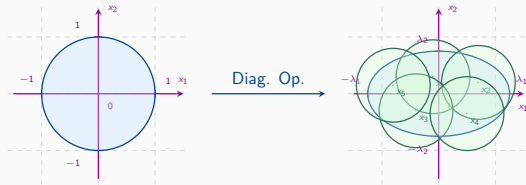
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$$H(\varepsilon) \simeq \log \left(\prod_{n=1}^d \frac{\lambda_n}{\varepsilon} \right)$$

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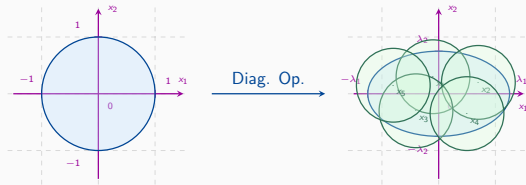
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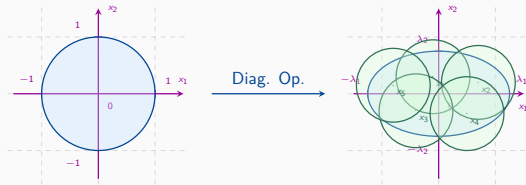
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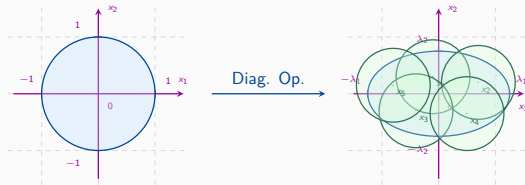
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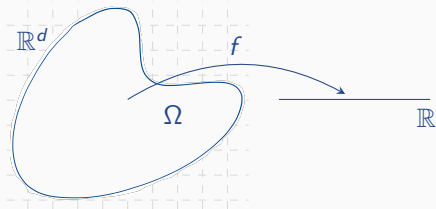
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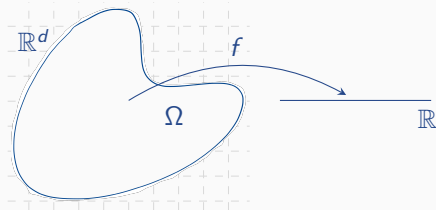
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Applications

Application to Sobolev Spaces on Bounded Domains

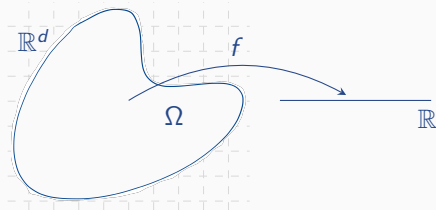


Application to Sobolev Spaces on Bounded Domains



- ▶ f is square-integrable
- ▶ the k -th derivatives of f are square-integrable
- ▶ f vanishes on the boundary $\partial\Omega$

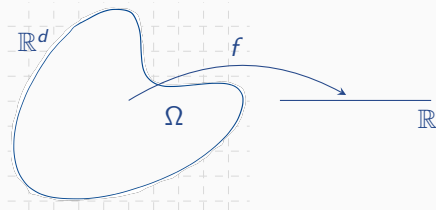
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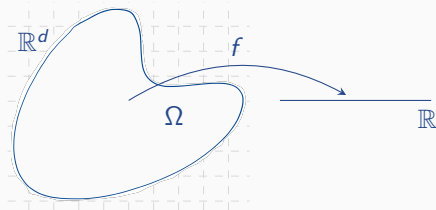


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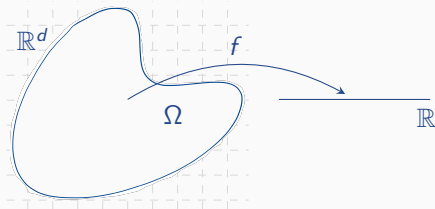
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In dimension 1 with $\Omega = [0, 2\pi]$:¹ $H(\varepsilon; \mathcal{B}_{[0,2\pi]}^{k,1}) \simeq 2k\varepsilon^{-\frac{1}{k}}$ In dimension d : $H(\varepsilon; \mathcal{B}_{\Omega}^{k,d}) \asymp \varepsilon^{-\frac{d}{k}}$.

¹D. L. Donoho, *Counting bits with Kolmogorov and Shannon*, Stanford University (2000) Tech. Rep. 2000-38.

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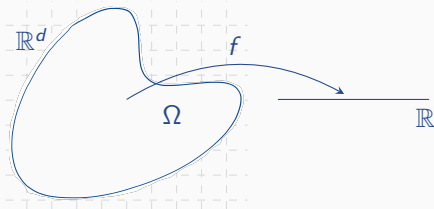
Theorem — A. & Bölcskei '25

For $d \geq 3$ and under a regularity condition on the boundary $\partial\Omega$

$$H(\varepsilon; \mathcal{B}_{\Omega}^{k,d}) = k\chi_d(\Omega)\varepsilon^{-\frac{d}{k}} + \frac{k\chi_{d-1}(\partial\Omega)}{4}\varepsilon^{-\frac{d-1}{k}} + o_{\varepsilon \rightarrow 0}(\varepsilon^{-\frac{d-1}{k}}) \quad \text{with} \quad \chi_r(S) := \frac{\omega_r}{r(2\pi)^r} \mathcal{H}^r(S)$$

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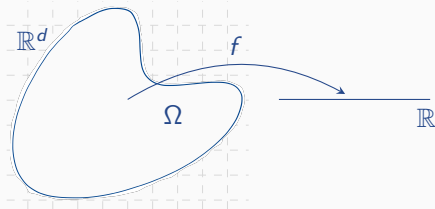
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Sobolev norm: $\|f\|_{\text{Sob.}}^2 = \|f\|_{L^2(\Omega)}^2 + \|f^{(k)}\|_{L^2(\Omega)}^2$ Unit ball: $\mathcal{B}_{\Omega}^{k,d} = \{\|f\|_{\text{Sob.}} \leq 1\} \subset L^2(\Omega)$

In dimension 1 with $\Omega = [0, 2\pi]$:¹ $H(\varepsilon; \mathcal{B}_{[0,2\pi]}^{k,1}) \simeq 2k\varepsilon^{-\frac{1}{k}}$ In dimension d : $H(\varepsilon; \mathcal{B}_{\Omega}^{k,d}) \asymp \varepsilon^{-\frac{d}{k}}$.

Theorem — A. & Bölcskei '25

For $d \geq 3$ and under a regularity condition on the boundary $\partial\Omega$

$$H(\varepsilon; \mathcal{B}_{\Omega}^{k,d}) = k\chi_d(\Omega)\varepsilon^{-\frac{d}{k}} + \frac{k\chi_{d-1}(\partial\Omega)}{4}\varepsilon^{-\frac{d-1}{k}} + o_{\varepsilon \rightarrow 0}(\varepsilon^{-\frac{d-1}{k}}) \quad \text{with} \quad \chi_r(S) := \frac{\omega_r}{r(2\pi)^r} \mathcal{H}^r(S)$$

¹D. L. Donoho, *Counting bits with Kolmogorov and Shannon*, Stanford University (2000) Tech. Rep. 2000-38.

Proof — Sobolev Spaces on Bounded Domains

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► [Link to Laplacian](#) $\|f\|_{\text{Sob.}}^2 = \langle f, f \rangle_{L^2(\Omega)} + \langle f, (-\Delta)^k f \rangle_{L^2(\Omega)} = \langle f, \overbrace{(\text{Id} + (-\Delta)^k)}^T f \rangle_{L^2(\Omega)}$

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$$\mathcal{B}_{\Omega}^{k,d} = T^{-1/2}(\mathcal{B}_{L^2(\Omega)}) \quad \text{so that} \quad H(\varepsilon; \mathcal{B}_{\Omega}^{k,d}) = H_{T^{-\frac{1}{2}}}(\varepsilon)$$

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► **Weyl law** for the Laplacian²

$$\lambda_n(-\Delta) \simeq \frac{4\pi^2}{(\omega_d \mathcal{H}^d(\Omega))^{\frac{2}{d}}} n^{\frac{2}{d}} + \frac{2\pi^2 \omega_{d-1} \mathcal{H}^{d-1}(\partial\Omega)}{d(\omega_d \mathcal{H}^d(\Omega))^{1+\frac{1}{d}}} n^{\frac{1}{d}}$$

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$$\lambda_n \simeq \frac{c_1}{n^{\alpha_1}} + \frac{c_2}{n^{\alpha_2}} \quad \text{implies} \quad H(\varepsilon) \simeq \alpha_1 \left(\frac{c_1}{\varepsilon} \right)^{\frac{1}{\alpha_1}} + \frac{(c_1/c_2)^{\frac{1-\alpha_2}{\alpha_1}}}{\alpha_1 - \alpha_2 + 1} \left(\frac{c_2}{\varepsilon} \right)^{\frac{1+\alpha_1-\alpha_2}{\alpha_1}}$$

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Remark. $\alpha_1 = k/d$ and $\alpha_2 = (k+1)/d$

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Remark. $\alpha_1 = k/d$ and $\alpha_2 = (k+1)/d \rightarrow \alpha_2 < \alpha_1 + 1/2$ is equivalent to $d > 2$.

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Outlook

$$M(\lambda) \simeq \kappa_1 \lambda^{-\beta_1} + \kappa_2 \lambda^{-\beta_2} \quad \text{implies} \quad H(\varepsilon) \simeq \frac{\kappa_1}{\beta_1} \varepsilon^{-\beta_1} + \frac{\kappa_2}{\beta_2} \varepsilon^{-\beta_2}$$

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► **Time-frequency representation**

$$Tf(x) = \mathcal{F}^{-1}\left(\sigma(x, \cdot) \mathcal{F}f\right)(x)$$

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$$Tf(x) = \mathcal{F}^{-1}\left(\sigma(x, \cdot) \mathcal{F}f\right)(x) \quad \text{then} \quad M(\lambda) \simeq \text{vol}\left(\left\{(x, \omega) \in \mathbb{R}^{2d} \mid \sigma(x, \omega) > \lambda\right\}\right)$$

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Sobolev spaces with potential on $\mathbb{R}^d$.

$$H(\varepsilon) \simeq \frac{k\omega_{2d}}{2d(2\pi\sqrt{c})^d} \varepsilon^{-2d/k}$$

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► **General formula** (under a weak regularity assumption on $M(\lambda)$)

$$H(\varepsilon) \simeq \int_{\varepsilon}^{\infty} \frac{M(\lambda)}{\lambda} d\lambda$$

works for **logarithmic**, **polynomial**, **poly-logarithmic**, **exponential** decay of eigenvalues

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T. Allard & H. Bölcskei, *Entropy and Minimax Risk of Hypocoelliptic Pseudodifferential Operators*, <https://arxiv.org/pdf/2603.23744>

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T. Allard, *Metric entropy and minimax risk of ellipsoids with an application to Pinsker's theorem*, <https://arxiv.org/pdf/2510.22441>

Thank You

Summary

Main Result

Question: Can we transfer asymptotics on the eigenvalues $\lambda_n(T)$ of T to its entropy?

Theorem — A. & Bölcskei '25

If $\lambda_n(T) = c_1 n^{-\alpha_1} + c_2 n^{-\alpha_2} + o_{n \rightarrow \infty}(n^{-\alpha_2})$ with $0 < \alpha_1 < \alpha_2 < \alpha_1 + 1/2$, then

$$H(\varepsilon) = \alpha_1 \left(\frac{c_1}{\varepsilon} \right)^{\frac{1}{\alpha_1}} + \frac{(c_1/c_2)^{\frac{1}{\alpha_1-1}}}{\alpha_1 - \alpha_2 + 1} \left(\frac{c_2}{\varepsilon} \right)^{\frac{1}{\alpha_1-1}} + o_{\varepsilon \rightarrow 0} \left(\varepsilon^{-\frac{1}{\alpha_1-1}} \right),$$

and $a_n = c_1 \alpha_1 n^{\alpha_1} m^{-\alpha_1} + \frac{c_1 c_2^{\frac{\alpha_1}{\alpha_1-1}}}{1 + \alpha_1 - \alpha_2} m^{-\alpha_2} + o_{m \rightarrow \infty}(m^{-\alpha_2})$

Reformulation. In terms of the **eigenvalue-counting function** $M(\lambda) = \#\{\lambda_n \geq \lambda\}$

If $M(\lambda) = \kappa_1 \lambda^{-\beta_1} + \kappa_2 \lambda^{-\beta_2} + o_{\lambda \rightarrow 0}(\lambda^{-\beta_2})$, with $0 < \beta_1/2 < \beta_2 < \beta_1$, then

$$H(\varepsilon) = \frac{\kappa_1}{\beta_1} \varepsilon^{-\beta_1} + \frac{\kappa_2}{\beta_2} \varepsilon^{-\beta_2} + o_{\varepsilon \rightarrow 0}(\varepsilon^{-\beta_2})$$

Proof Sketch

$$\lambda_n(T) \simeq \frac{c_1}{n^{\alpha_1}} + \frac{c_2}{n^{\alpha_2}} \text{ implies } H(\varepsilon) \simeq \alpha_1 \left(\frac{c_1}{\varepsilon} \right)^{\frac{1}{\alpha_1}} + \frac{(c_1/c_2)^{\frac{1}{\alpha_1-1}}}{\alpha_1 - \alpha_2 + 1} \left(\frac{c_2}{\varepsilon} \right)^{\frac{1}{\alpha_1-1}}$$



Log Op



Effective Dimension: $\lambda_d \simeq \varepsilon$

Lower Bound: Volume Arguments

Upper Bound: Block Decomposition + Rogers' Bound

$$H(\varepsilon) \simeq \log \left(\prod_{n=1}^d \frac{\lambda_n}{\varepsilon} \right) \stackrel{\lambda_n \simeq \varepsilon}{\simeq} \log \left(\prod_{n=1}^d \frac{\lambda_n}{\lambda_d} \right) \simeq \alpha_1 \log \left(\frac{d^{\alpha_1}}{d!} \right) \stackrel{\text{Stirling}}{\simeq} \alpha_1 d^{\frac{\alpha_1-1}{2}} \simeq \alpha_1 d^{\frac{\alpha_1-1}{2}} \left(\frac{c_1}{\varepsilon} \right)^{\frac{1}{\alpha_1}}$$

Application to Sobolev Spaces on Bounded Domains



► f is square-integrable

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► f vanishes on the boundary $\partial\Omega$

Sobolev norm: $\|f\|_{B_{0,k}^s}^2 = \|f\|_{L^2(\Omega)}^2 + \|f^{(k)}\|_{L^2(\Omega)}^2$ **Unit ball:** $B_{0,k}^s = \{f : \|f\|_{B_{0,k}^s} \leq 1\} \subset L^2(\Omega)$

In dimension 1 with $\Omega = [0, 2\pi]^{-1}$ $H(\varepsilon; B_{0,k}^{k,1}) \simeq 2k\varepsilon^{-\frac{1}{2}}$ In dimension d : $H(\varepsilon; B_{0,k}^{k,d}) \asymp \varepsilon^{-\frac{d}{2}}$

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²A. Ibrat & W. S. Rabin, Entropy and Minimum Risk of Hypothesis Pseudodifferential Operators, <https://arxiv.org/abs/2003.12044>

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³T. Koenig, Entropy and minimum risk of hypothesis with an application to Fisher's theorem, <https://arxiv.org/abs/2003.12044>



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